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On a Conjecture of P. Landrock

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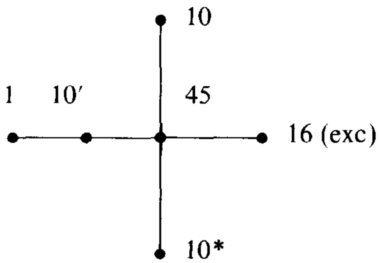
Let G be a finite group, k an algebraically closed field of prime characteristic p and let J be the radical of the group ring kG. Landrock conjectured in his book [4] that

$$(L):\quad J^i/J^{i+1} \text{ is self-dual as a (right) } kG\text{-module for all } i.$$

Unfortunately this is not true in general. A counterexample will be presented. However, we acknowledge the significance of this conjecture for various reasons. For example it will be helpful in determining Loewy series for given groups. So it seems to be reasonable to investigate when or for which groups it is true. We would like to do this in the second section.

1. COUNTEREXAMPLES

We show that (L) is false for the Mathieu group $G = M_{11}$ of degree eleven and $p = 11$. G has a Sylow 11-subgroup of order 11 and the Brauer tree of its principal 11-block B is



Therefore $I\,\mathrm{Br}(B) = \{1, 9, 10, 10^*, 16\}$ and 10^* is the dual of 10 (see James [1]). Let P_i be the projective cover of $i \in I\,\mathrm{Br}(B)$. It is not so difficult

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to find Loewy series of P_i by making use of following facts for blocks with cyclic defect groups:

(1) $\dim \operatorname{Ext}^1_{kG}(i, j) = \dim \operatorname{Ext}^1_{kG}(j^*, i^*) \leqslant 1$;

(2) each Loewy factor of P_i consists of at most two simple components (Peacock [5]);

(3) P_1, P_{10}, P_{10^*} are uniserial, while others are not (Janusz [2]); and possibly by making use of the general fact:

(4) if V is a simple component of P_iJ^r/P_iJ^{r+1}, then there exists a simple component U of P_iJ^{r-1}/P_iJ^r such that $\operatorname{Ext}^1_{kG}(U, V) \neq 0$ (see Landrock [4, p. 38])

Thus the Loewy series are as follows;

	1		10		10*		9		16
P_1:	9	P_{10}:	9	P_{10^*}:	16	P_9:	1 10*	P_{16}:	10 16
	1		10*		10		16		9
			16		9		10		10*
			10		10*		9		16

This implies that in J/J^2 10 appears with multiplicity 16, while 10* with multiplicity 9 and hence J/J^2 is not self-dual.

Next, we give counter examples in solvable groups; Let p be an odd prime and $q = p^p$. Let G be a permutation group on the Galois field $\mathrm{GF}(q)$ consisting of the transformations of the form

$$(\sigma, a, b): \quad v \to av^\sigma + b$$

with $a \in GF(q)^\times = \mathrm{GF}(q) - \{0\}$, $b \in GF(q)$ and $\sigma \in \mathrm{Gal}(\mathrm{GF}(q)/F)$, where $F = \mathrm{GF}(p)$. Put $P = \{(1, 1, b); b \in \mathrm{GF}(q)\}$, $M = \{(1, a, 0); a \in \mathrm{GF}(q)^\times\}$ and let $\tau \in \mathrm{Gal}(\mathrm{GF}(q)/F)$ be such that $v^\tau = v^p$ for $v \in \mathrm{GF}(q)$. Then P is a normal p-subgroup of G and M is a p'-subgroup of G which is isomorphic to $\mathrm{GF}(q)^\times$. Let x be a generator of M. G has a normal series $G \rhd MP \rhd P \rhd 1$ and $[\tau, M] = \langle x^{p-1} \rangle$, $[G : G'] = (p-1)p$. P is an FG-module by the conjugation of the elements of G, which is irreducible. In fact P_M is irreducible as M acts on $P^\times$ transitively. So that the characteristic polynomial of x on P is irreducible and the roots of it are $\zeta, \zeta^p, ..., \zeta^{p^{p-1}}$, where ζ denotes a primitive $(q-1)$th root of unity in k. This means that P is absolutely irreducible. We set $V = k \otimes_F P$. The irreducible kG-modules are either one-dimensional or p-dimensional and there are exactly $(p-1)$ one-dimensional kG-modules, which are denoted by $S_1 = k$, $S_2, ..., S_{p-1}$. Since $\theta = \zeta^{q-1/p-1}$ is a primitive $(p-1)$th root of unity, we may assume that the character value of S_i at x is a θ^{i-1}. Let U_i be te projective cover of S_i. Note that $U_i \simeq U_1 \otimes_k S_i$. Moreover we have that $U_1J/U_1J^2 \cong k \oplus V$, which follows from a result of Gaschütz (see Schmidt [6]) and $U_iJ/U_iJ^2 \simeq S_i \oplus S_i \otimes_k V$. We claim that no $V \otimes_k S_i$ is isomorphic to V^*. In fact if $V \otimes_k S_i \simeq V^*$ for some i, then by considering the characteristic roots of the action of x on both modules, we get that $\zeta\theta^{i-1} = \zeta^{-p^j}$ for some j $(0 \leqslant j \leqslant p-1)$. This is clearly impossible as p is odd. It is now easy to see that V appears in J/J^2 with multiplicity 1 mod p, while V^* with 0 mod p. Therefore J/J^2 is not self-dual.

The above argument will be valid for $p=2$ by considering similar transformation group on $\mathrm{GF}(2^4)$ instead.

2. Some Affirmative Cases

We put $A = kG$. For a right module V over A, V^* denotes the dual space $\mathrm{Hom}_k(V, k)$, which is again a right module over A by the rule $(fx)(v) = f(vx^{-1})$ for $f \in V^*$, $x \in G$, and $v \in V$. The conjecture (L) is equivalent to saying that J^i/J^{i+1} is isomorphic to $\mathrm{soc}_{i+1}(A)/\mathrm{soc}_i(A)$, since the former is isomorphic to the dual of the latter in general. This nice relation has motivated the present work and we want to investigate when or for which groups the conjecture is true. However, it is almost impossible to go into direct discussion upon the structure of J^i, which will be far from being handled at present. We proceed by finding conditions for the validity of (L), which may be of use to some extent (see the Lemma 3 and the Corollary 5).

We denote by $pi(A)$ the set of primitive idempotents of A and by δ the anti-automorphism of A defined as $\delta(\sum_{x \in G} a_x x) = \sum_{x \in G} a_x x^{-1}$, where $a_x \in k$. The following fact is easy to show, but useful.

Lemma 1. *For $e \in pi(A)$, $\delta(e)A \simeq (eA)^*$ as A-modules.*

Proof. If T is a k-representation afforded by $V = eA/eJ$, then $U(x) = {}^tT(x^{-1})$ is the k-representation of G afforded by V^*. Therefore $\mathrm{tr}\, U(\delta(e)) = \mathrm{tr}\, T(e) = 1$ and hence $\delta(e)A/\delta(e)J \simeq (eA/eJ)^* \simeq \mathrm{soc}(eA)^*$. This implies that $\delta(e)A \simeq (eA)^*$. ▮

We mention two applications of Lemma 1. The first one is to give short proofs to the theorems of Landrock [3], which is another purpose of this paper. For simple A-modules U, V, let $c^{(s)}(U, V)$ be the Cartan integer corresponding to U and V in the factor algebra A/J^s. So if $e, f \in pi(A)$ such that $eA/eJ \simeq U$, $fA/fJ \simeq V$, then this number equals to dim eAf/eJ^sf. Let f^* be a primitive idempotent of A such that $f^*A \simeq (fA)^*$. Under the above notation, we have

THEOREM 2 (Landrock). (1) *The multiplicity of V as a simple component of eJ^{s-1}/eJ^s equals that of U^* in f^*J^{s-1}/f^*J^s.*

(2) $c^{(s)}(U, V) = c^{(s)}(V^*, U^*)$.

Proof. $eJ^{s-1}f/eJ^sf \simeq \delta(f)\, J^{s-1}\delta(e)/\delta(f)\, J^s\delta(e) \simeq f^*J^{s-1}e^*/f^*J^se^*$, since $f^*A \simeq \delta(f)A$ by Lemma 1. This proves the first assertion. The second one can be shown quite similarly. ▮

Also we have

LEMMA 3. *(L) is true if and only if* $\dim J^ie = \dim eJ^i$ *for all i and all $e \in pi(A)$.*

Proof. We see that $(L) \Leftrightarrow \dim J^ie/J^{i+1}e = \dim J^i\,\delta(e)/J^{i+1}\,\delta(e) \Leftrightarrow \dim J^ie = \dim J^i\,\delta(e)$ for all i and all $e \in pi(A)$. The first equivalence is clear. To show $(\Rightarrow)$ in the second one, we proceed with the induction on i. Note that in the symmetric algebra A, $Ia = 0 \Leftrightarrow aI = 0$ for any two-sided ideal I of A, as is easily seen from the existence of a symmetric linear function on A which does not annihilate any one-sided ideal of A. From this we see that if $J^ne = (\mathrm{soc}\, A)e$, then $eJ^n = e(\mathrm{soc}\, A)$. By applying δ, we get $J^n\,\delta(e) = (\mathrm{soc}\, A)\,\delta(e)$. Since $\mathrm{soc}\, A$ is self-dual, we have $\dim J^ne = \dim (\mathrm{soc}\, A)e = \dim(\mathrm{soc}\, A)\,\delta(e) = \dim J^n\,\delta(e)$. It then follows from the assumption that $\dim J^{n-1}e = \dim J^{n-1}\,\delta(e) = \dim eJ^{n-1}$. Therefore we have $\dim J^ie = \dim eJ^i$ for all i by the induction. ▮

In the ordinary case, every irreducible character is algebraically conjugate to its dual. However, in the modular case, this is not true in general.

PROPOSITION 4. *If every irreducible k-character is algebraically conjugate to its dual, then (L) is true.*

Proof. Let F be the prime field of k and let I be the radical of FG. We have $J = kI$ as is well known and so $J^i/J^{i+1} \simeq k \otimes_F I^i/I^{i+1}$. If V is a simple FG-module, then $k \otimes_F V$ is a direct sum of nonisomorphic simple kG-modules which are algebraically conjugate to each other. Moreover if U is a simple FG-module not isomorphic to V, then $k \otimes_F V$ and $k \otimes_F U$ do not have any simple component in common. Putting together these things, we see that two algebraically conjugate simple kG-modules appear with the same multiplicity in J^i/J^{i+1}. Therefore our assertion follows by the assumption. ▮

COROLLARY 5. *Let $|G| = p^am$ with $(p, m) = 1$. Then (L) is true if there is an integer n such that $p^n \equiv -1 \bmod m$.*

Proof. The assumption is equivalent to that a primitive mth root of unity in k is conjugate to its inverse. So, in particular, every irreducible k-character is algebraically conjugate to its dual. ∎

For a p-group G, (L) is trivially true. So it is natural to expect that (L) is true if G has a normal Sylow p-subgroup. More precisely we have

LEMMA 6. *Let $G \trianglerighteq H$ and assume that $(p, [G:H]) = 1$. Then (L) is true for G if and only if it is true for H.*

Proof. Let L be the radical of kH. It follows from the assumption that $J^i = L^iA = AL^i$ for all i. So if $L^i/L^{i+1} \simeq (L^i/L^{i+1})^*$, we have $J^i/J^{i+1} \simeq L^i/L^{i+1} \otimes_H kG \simeq (L^i/L^{i+1})^* \otimes_H kG \simeq (J^i/J^{i+1})^*$. To show the converse, let $\{W_1, W_2, \ldots, W_s\}$ be a full set of nonisomorphic simple kH-modules and let $\{\Delta; \Delta \in \Gamma\}$ be the set of G-orbits of $\{W_1, W_2, \ldots, W_s\}$ under the conjugate action of G. We set, for fixed i, $L^i/L^{i+1} \simeq \bigoplus_j a_j W_j$, where $a_j \geqslant 0$. We want to show $a_j = a_{j^*}$ for all j, where $W_{j^*} \simeq W_j^*$. Put $W^G = W \otimes_H kG$. Recall that W_q^G and W_r^G have simple components in common if and only if W_q and W_r are G-conjugate. In that case we have $a_q = a_r$, which is because of that

(P): L^i is isomorphic to its G-conjugate, as $L^i = x^{-1}Lx \simeq L^ix$ for $x \in G$.

Summarizing the above, we have $J^i/J^{i+1} \simeq \bigoplus_{\Delta \in \Gamma} a_\Delta |\Delta|\, W_\Delta^G$, where if $W_\Delta \simeq W_j \in \Delta$, then $a_\Delta = a_j$. And hence $(J^i/J^{i+1})^* \simeq \bigoplus_\Delta a_\Delta |\Delta|\, W_{\Delta^*}^G$. Since $(J^i/J^{i+1})^* \simeq (J^i/J^{i+1}) \simeq \bigoplus_\Delta a_{\Delta^*} |\Delta^*|\, W_{\Delta^*}^G$ by the assumption, we get $a_\Delta |\Delta| = a_{\Delta^*} |\Delta^*|$ and thus $a_\Delta = a_{\Delta^*}$ as asserted. ∎

In the rest we show that (L) is true for particular groups.

PROPOSITION 7. *(L) is true for A_n.*

Proof. (L) is true for S_n, so that we may assume that $p = 2$ by Lemma 6. Let $x = (12)$ and let φ be arbitrary irreducible k-character of A_n. By (P) it suffices to show that $\varphi^* = \varphi^x$ unless $\varphi^* = \varphi$. If $\varphi^x = \varphi$, then φ extends to S_n and hence $\varphi^* = \varphi$. If $\varphi^x \neq \varphi$, there exists an irreducible k-character ψ of S_n such that $\psi = \varphi + \varphi^x$ on A_n. Therefore $\psi = \psi^* = \varphi + \varphi^*$ on A_n, whence it follows that $\varphi^x = \varphi^*$. ∎

One may note from Lemma 3 that (L) is true if there is an anti-automorphism μ of A such that $\mu(e)A \simeq eA$ for any $e \in pi(A)$. This is just the case for $G = GL_n(q)$ or $U_n(q)$. To see this, let $\mu(\sum_{x \in G} a_x x) = \sum_{x \in G} a_x^t x$, where ${}^t x$ denotes the transpose of x. This is an anti-automorphism of A. On the other hand we know that $x \in G$ is conjugate to ${}^t x$ in either case (see Wall [7]. Of course for $GL_n(q)$, this is easily seen from elementary linear algebra). Let $e \in pi(A)$ and let T be an irreducible k-representation of G afforded by eA/eJ. It is easy to see that tr $T(\mu(e)) =$ tr $T(e) = 1$, whence it follows that $\mu(e)A \simeq eA$. Thus we have shown

PROPOSITION 8. *(L) is true for $GL_n(q)$ and $U_n(q)$.*

Finally we show

PROPOSITION 9. *(L) is true for $SL_n(q)$.*

Proof. There exists $GL_n(q) \rhd X \rhd SL_n(q)$, such that $GL_n(q)/X$ is a p-group and $X/SL_n(q)$ is a p'-group and X is closed under the transpose action. By Lemma 6, it is sufficient to show that (L) is true for X. Put $B = k[GL_n(q)]$, $B_0 = kX$ and let $e \in pi(B_0)$. Since $GL_n(q)/X$ is a p-group, $eB \simeq eB_0 \otimes_X B$ is indecomposable (projective) by Green's theorem. Therefore $eB \simeq \mu(e)B$ as remarked above. It then follows that eB_0 is isomorphic to a $GL_n(q)$-conjugate of $\mu(e)\,B_0$; $eB_0 \simeq \mu(e)\,B_0x$ for some $x \in B$. Let L be the radical of B_0. From the above isomorphism it follows that $eL^i \simeq \mu(e)\,B_0xL^i = \mu(e)\,L^ix$ and dim $eL^i =$ dm $\mu(e)\,L^i =$ dim L^ie, which implies our assertion by Lemma 3. ∎

Remark. (L) remains true for $PGL_n(q)$, $PU_n(q)$, and $PSL_n(q)$, as is easily seen from the above argument.

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